

Spectral Condition for a Graph to be Hamiltonian with respect to Normalized Laplacian

Yi-Zheng Fan^{1,*}, Gui-Dong Yu^{1,2,†}

1. School of Mathematical Sciences, Anhui University, Hefei 230601, P.R. China

2. School of Mathematics & Computation Sciences, Anqing Normal College, Anqing 246011, P.R. China

Abstract: Let G be a graph and let Δ, δ be the maximum and minimum degrees of G respectively, where $\Delta/\delta < c < \sqrt{2}$ and c is a constant. In this paper we establish a sufficient spectral condition for the graph G to be Hamiltonian, that is, the nontrivial eigenvalues of the normalized Laplacian of G are sufficiently close to 1.

Keywords: Graph; Hamiltonian; normalized Laplacian

MR Subject Classifications: 05C45, 05C50

1 Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. The *adjacency matrix* of G is defined to be a matrix $A = [a_{ij}]$ of order n , where $a_{ij} = 1$ if v_i is adjacent to v_j , and $a_{ij} = 0$ otherwise. Let D be the diagonal matrix of order n whose (i, i) -entry is d_{v_i} , the degree of the vertex v_i of G . The *signless Laplacian*, the *Laplacian*, and the *normalized Laplacian* of G are respectively defined by $Q = D + A$, $L = D - A$ and $\mathcal{L} = D^{-1/2}LD^{-1/2}$ (for the last matrix we assume the graph contains no isolated vertices).

The graph G is said to be *Hamiltonian* if there exists a cycle passing through all the vertices of G . Such cycle is called a *Hamiltonian cycle* of G . The question of deciding whether or not a given graph is Hamiltonian is a very difficult one; indeed, determining whether a given graph

*Corresponding author. Email: fanyz@ahu.edu.cn. Supported by National Natural Science Foundation of China (11071002), Program for New Century Excellent Talents in University, Key Project of Chinese Ministry of Education (210091), Specialized Research Fund for the Doctoral Program of Higher Education (20103401110002), Science and Technological Fund of Anhui Province for Outstanding Youth (10040606Y33), Scientific Research Fund for Fostering Distinguished Young Scholars of Anhui University (KJJQ1001), Project for Academic Innovation Team of Anhui University (KJTD001B).

†Email: yuguid@aqtc.edu.cn. Supported by NSF of Department of Education of Anhui Province (KJ2011A195) and Innovation Fund for Graduates of Anhui University.

is Hamiltonian is NP-complete [4]. Recently the spectral graph theory has been applied to this problem. The sufficient spectral conditions are given for a graph having Hamiltonian paths or Hamiltonian cycles or being Hamilton-connected, in terms of spectral radius of a graph or its complement, with respect to the adjacency matrix or Laplacian or signless Laplacian; see Fiedler and Nikiforov [3], Zhou [10], Yu and Fan [11]. However, these conditions always imply the graph are very dense.

A breakthrough in studying Hamiltonicity occurred in 1975 when Komlós and Szemerédi [5] showed that almost surely every random graph is Hamiltonian. The technique involves the rotation of paths attributed to Posa [7]. Krivelevich and Sudakov [6] established a sufficient condition for a d -regular graph to be Hamiltonian. They showed that if σ , the second largest absolute value of an eigenvalue of the adjacency matrix of a d -regular graph, satisfies

$$\sigma \leq c \frac{(\log \log n)^2}{\log n (\log \log \log n)} d, \quad (1.1)$$

for a constant c and n sufficiently large, then G is Hamiltonian. The condition (1.1) is not based on density conditions, rather it implies the graph is pseudo-random (the edge distribution resembles closely that of a truly random graph $G(n, d/n)$).

Using Laplacian of graphs, Butler and Chung [1] established a sufficient condition for a graph G being Hamiltonian. They proved that if

$$|d - \mu_i| \leq c \frac{(\log \log n)^2}{\log n (\log \log \log n)} d, \quad (1.2)$$

for $i \neq 0$, some constant c and n sufficiently large, then G is Hamiltonian, where d is the average degree of G , and $0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_{n-1}$ are the eigenvalues of the Laplacian of G . The condition (1.2) implies the graph G is almost regular, and in fact, pseudo-random. If G is regular, then (1.2) is exactly (1.1).

Mary Radcliffe [8] promoted the problem of finding sufficient conditions on the spectrum of the normalized Laplacian to ensure that a graph is Hamiltonian. In this paper, we regard this problem and get the following result. It can be seen the result also implies that of Krivelevich and Sudakov for regular graphs.

THEOREM 1.1 *Let G be a graph on n vertices, $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ be the eigenvalues of the normalized Laplacian of G . Assume that $\Delta/\delta < c < \sqrt{2}$ for some constant c , where Δ, δ are the maximum and minimum degrees of the vertices of G . If*

$$|1 - \lambda_i| \leq \frac{(\log \log n)^2}{7500 \log n (\log \log \log n)}, \quad (1.3)$$

for $i \neq 0$ and n sufficiently large, then G is Hamiltonian.

Remark: We show two points on Theorem 1.1 by an example. Let G be the graph obtained from a complete graph K_{n-1} on $n-1$ vertices by joining a new vertex with $\beta := \lceil \alpha(n-1) \rceil$ vertices of K_{n-1} , where $0 < \alpha < 1$. It is not too hard to show that:

$$\max_{i \neq 0} |1 - \lambda_i| = \frac{n-2+\beta + \sqrt{(n-2+\beta)^2 + 4(n-1)(n-2)(n-\beta-2)}}{2(n-1)(n-2)} \approx \frac{\sqrt{1-\alpha}}{\sqrt{n}}.$$

So, this graph has the eigenvalues very tightly clustered near 1 (i.e., even tighter than the bound in (1.3)).

(1) The constraint in Theorem 1.1 on the ratio of the maximal degree and minimal degree is necessary. If taking $\alpha = \frac{1}{n-1}$, i.e., G is K_{n-1} with a pendant edge, surely G is not Hamiltonian. In this case $\frac{\Delta}{\delta} = n-1 \rightarrow \infty$.

(2) Theorem 1.1 applies more Hamiltonian graphs than Butler and Chung's result. The condition (1.2) (or see Theorem 2.1 of [1]) implies that $-\frac{1}{n} - \epsilon \leq \frac{d_v}{d} - 1 \leq \epsilon$ for each vertex v , where $\epsilon = c \frac{(\log \log n)^2}{\log n (\log \log \log n)}$. So, when n goes to infinity, $\frac{d_v}{d} - 1 \rightarrow 0$, which implies the graph is almost regular.

For the above graph G , if taking α being a constant such that $\sqrt{2}/2 < \alpha < 1$, then $\frac{\Delta}{\delta} < \sqrt{2}$. Surely G is Hamiltonian, which is consistent with our result. However, $|\frac{\delta}{d} - 1| \rightarrow 1 - \alpha \neq 0$. So, using Butler and Chung's condition, we cannot decide whether it is Hamiltonian or not.

2 Preliminaries

Let G be a graph, and let $X \subset V(G)$. Denote by \bar{X} be the complement of X in $V(G)$, and by $N(X)$ the set of all vertices in $V \setminus X$ adjacent to some vertex in X . The *volume* of X , denoted by $\text{vol}(X)$, is defined as $\text{vol}(X) = \sum_{v \in X} d_v$. The volume of G is denoted by $\text{vol}(G) = \sum_{v \in G} d_v$. For two subsets X and Y of V , we let $e(X, Y)$ be the number of edges with one endpoint in X and one in Y , while $e(X)$ be the number of edges with both endpoints in X .

THEOREM 2.1 [2] *Let G be a graph on n vertices, and let the eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ of the normalized Laplacian of G satisfy $|1 - \lambda_i| \leq \lambda$ for $i \neq 0$. Then for any two subsets X and Y of the vertices in G ,*

$$\left| e(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \leq \lambda \frac{\sqrt{\text{vol}(X)\text{vol}(\bar{X})\text{vol}(Y)\text{vol}(\bar{Y})}}{\text{vol}(G)}.$$

By Theorem 2.1, we have the following conclusion immediately in terms of the maximum and minimum degrees.

COROLLARY 2.2 *Let G be a graph on n vertices with average degree d , and let the eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ of the normalized Laplacian of G satisfy $|1 - \lambda_i| \leq \lambda$ for $i \neq 0$. Then for*

any two subsets X and Y of the vertices in G ,

$$\begin{aligned} e(X, Y) &\geq \frac{\delta^2}{nd} |X||Y| - \frac{\lambda\Delta^2}{nd} \sqrt{|X|(n-|X|)|Y|(n-|Y|)}, \\ e(X, Y) &\leq \frac{\Delta^2}{nd} \left(|X||Y| + \lambda\sqrt{|X|(n-|X|)|Y|(n-|Y|)} \right). \end{aligned}$$

If we consider the case $X = \{v\}$ and $Y = V \setminus \{v\}$, then Corollary 2.2 implies that

$$\frac{n-1}{n} \frac{\delta^2}{d} - \frac{\Delta^2}{d} \lambda \leq d_v \leq \frac{\Delta^2}{d} (1 + \lambda).$$

COROLLARY 2.3 *Let G be a graph on n vertices with average degree d , and let the eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ of the normalized Laplacian of G satisfy $|1 - \lambda_i| \leq \lambda$ for $i \neq 0$. Then for any subset X of the vertices in G ,*

$$\frac{\delta^2}{2nd} |X|(|X| - 1) - \frac{\lambda\Delta^2}{nd} |X|(n - |X|/2) \leq e(X) \leq \frac{\Delta^2}{2nd} (|X|(|X| - 1) + 2\lambda|X|(n - |X|/2)).$$

Proof: Let $x = |X|$, $X' \subset X$ and $|X'| = \lfloor x/2 \rfloor = x'$. Since

$$\sum_{\substack{X' \subset X \\ |X'| = x'}} e(X', X \setminus X') = \binom{x-2}{x'-1} e(X, X),$$

by the upper bound of $e(X, Y)$ in Corollary 2.2, we have

$$\begin{aligned} \binom{x-2}{x'-1} e(X, X) &= \sum_{\substack{X' \subset X \\ |X'| = x'}} e(X', X \setminus X') \\ &\leq \sum_{\substack{X' \subset X \\ |X'| = x'}} \frac{\Delta^2}{nd} \left(|X'||X \setminus X'| + \lambda\sqrt{|X'|(n-|X'|)|X \setminus X'|(n-|X \setminus X'|)} \right) \\ &= \binom{x}{x'} \frac{\Delta^2}{nd} \left(x'(x-x') + \lambda\sqrt{x'(n-x')(x-x')(n-x+x')} \right). \end{aligned}$$

So

$$\begin{aligned} e(X) &= \frac{1}{2} e(X, X) \\ &\leq \binom{x}{x'} \binom{x-2}{x'-1}^{-1} \frac{\Delta^2}{2nd} \left(x'(x-x') + \lambda\sqrt{x'(n-x')(x-x')(n-x+x')} \right) \\ &\leq \frac{\Delta^2}{2nd} \left[x(x-1) + 2\lambda x \left(n - \frac{x}{2} \right) \right]. \end{aligned}$$

Similarly, by the lower bound of $e(X, Y)$ in Corollary 2.2, we have

$$\begin{aligned} e(X) &= \frac{1}{2} e(X, X) \\ &\geq \binom{x}{x'} \binom{x-2}{x'-1}^{-1} \left(\frac{\delta^2}{2nd} x'(x-x') - \frac{\lambda\Delta^2}{2nd} \sqrt{x'(n-x')(x-x')(n-x+x')} \right) \\ &\geq \frac{\delta^2}{2nd} x(x-1) - \frac{\lambda\Delta^2}{nd} x \left(n - \frac{x}{2} \right). \end{aligned}$$

■

COROLLARY 2.4 *Let G be a graph on n vertices with average degree d , and let the eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ of the normalized Laplacian of G satisfy $|1 - \lambda_i| \leq \lambda$ for $i \neq 0$. Further assume that $\lambda < 1/8$, $(\Delta/\delta)^2 \leq 2(n-1)/n$, and that $X, Y \subseteq V$. Then the following results hold:*

- (a) *if $|X| < \lambda n$, then $e(X) \leq \frac{3\lambda\Delta^2}{2d}|X|$;*
- (b) *if $|X| < \lambda^2 n$, then $|N(X)| > \frac{(\frac{1}{2} - 4\lambda)^2}{3\lambda^2}|X|$;*
- (c) *if $|X| > \lambda^2 \Delta^4 n / \delta^4$, then $|N(X)| > \frac{n}{2} - |X|$;*
- (d) *if $X \cap Y = \emptyset$ and $e(X, Y) = 0$, then $|X||Y| < \lambda^2 \Delta^4 n^2 / \delta^4$;*
- (e) *G is connected.*

Proof: For (a) we use Corollary 2.3 and the assumption to get

$$e(X) \leq \frac{\Delta^2}{2nd}(|X|(|X| - 1) + 2\lambda|X|(n - |X|/2)) \leq \frac{\Delta^2}{2nd}(\lambda n|X| + 2\lambda n|X|) = \frac{3\lambda\Delta^2}{2d}|X|.$$

For (b) if $|X| < \lambda^2 n$, then $|X| < \lambda n$. We use (a), i.e., $e(X) \leq \frac{3\lambda\Delta^2}{2d}|X|$, and the remark following Corollary 2.2,

$$e(X, N(X)) = \sum_{x \in X} d_x - 2e(X) \geq \left(\frac{n-1}{n} \frac{\delta^2}{d} - \frac{\lambda\Delta^2}{d} \right) |X| - \frac{3\lambda\Delta^2}{d}|X| = \left(\frac{n-1}{n} \frac{\delta^2}{d} - \frac{4\lambda\Delta^2}{d} \right) |X|. \quad (2.1)$$

On the other hand, by Corollary 2.2,

$$\begin{aligned} e(X, N(X)) &\leq \frac{\Delta^2}{nd} \left(|X||N(X)| + \lambda \sqrt{|X|(n - |X|)|N(X)|(n - |N(X)|)} \right) \\ &\leq \frac{\Delta^2}{nd} |X||N(X)| + \frac{\lambda\Delta^2}{d} \sqrt{|X||N(X)|}. \end{aligned}$$

If $|N(X)| \leq \frac{(\frac{1}{2} - 4\lambda)^2}{3\lambda^2}|X|$ then we would have

$$\begin{aligned} \frac{\Delta^2}{nd} |X||N(X)| + \frac{\Delta^2}{d} \lambda \sqrt{|X||N(X)|} &\leq \frac{\Delta^2}{nd} \frac{(\frac{1}{2} - 4\lambda)^2}{3\lambda^2} |X|^2 + \frac{\Delta^2}{d} \frac{\lambda(\frac{1}{2} - 4\lambda)}{\sqrt{3}\lambda} |X| \\ &< \frac{\Delta^2}{d} \frac{(\frac{1}{2} - 4\lambda)^2}{3} |X| + \frac{\Delta^2}{d} \frac{\frac{1}{2} - 4\lambda}{3/2} |X| \\ &\leq \frac{\Delta^2}{d} \left(\frac{1}{2} - 4\lambda \right) |X|, \end{aligned}$$

using that $\lambda < 1/8$, $(\Delta/\delta)^2 \leq 2(n-1)/n$ in going to the last line, which is contradiction to (2.1), establishing (b).

For (c) letting $Y = V \setminus (X \cup N(X))$ and using Corollary 2.2, we have

$$\begin{aligned} 0 = e(X, Y) &\geq \frac{\delta^2}{nd} |X||Y| - \lambda \frac{\Delta^2}{nd} \sqrt{|X|(n - |X|)|Y|(n - |Y|)} \\ &\geq \frac{\delta^2}{nd} |X||Y| - \lambda \frac{\Delta^2}{d} \sqrt{|X||Y|(1 - \frac{|Y|}{n})}, \end{aligned}$$

which upon rearranging gives

$$\frac{|Y|}{1 - |Y|/n} \leq \frac{\lambda^2 \Delta^4 n^2}{\delta^4 |X|} < n.$$

This implies that $|Y| < n/2$ and hence $|N(X)| = n - |X| - |Y| > \frac{n}{2} - |X|$.

For (d) again using Corollary 2.2, we have

$$\begin{aligned} 0 = e(X, Y) &\geq \frac{\delta^2}{nd} |X||Y| - \lambda \frac{\Delta^2}{nd} \sqrt{|X|(n - |X|)|Y|(n - |Y|)} \\ &> \frac{\delta^2}{nd} |X||Y| - \lambda \frac{\Delta^2}{d} \sqrt{|X||Y|}; \end{aligned}$$

and the result follows.

For (e), if G is disconnected then G has a connected component X of size $|X| \leq n/2$. Since $|N(X)| = \emptyset$, it follows from part (c) that $|X| \leq \lambda^2 \Delta^4 n / \delta^4 \leq \frac{1}{8} \frac{4(n-1)^2}{n^2} \lambda n < \lambda n$. We use (a), i.e., $e(X) \leq \frac{3\lambda\Delta^2}{2d} |X|$, and the remark following Corollary 2.2,

$$\begin{aligned} e(X, N(X)) &= \sum_{x \in X} d_x - 2e(X) \\ &\geq \left(\frac{n-1}{n} \frac{\delta^2}{d} - \frac{\lambda\Delta^2}{d} \right) |X| - \frac{3\lambda\Delta^2}{d} |X| \\ &= \left(\frac{n-1}{n} \frac{\delta^2}{d} - \frac{4\lambda\Delta^2}{d} \right) |X| \\ &> \left(\frac{n-1}{n} \frac{\Delta^2 n}{2(n-1)d} - \frac{\Delta^2}{2d} \right) |X| = 0, \end{aligned}$$

a contradiction. ■

3 Proof of Theorem 1.1

The idea of the proof of Theorem 1.1 is to find a maximal path that can be closed to create a cycle. Using the assumptions and Corollary 2.4, G is connected, which implies that G is Hamiltonian (if not, there would be a vertex adjacent to some vertex in the cycle, allowing us to create a path of longer length). The technique used here is the rotation of the paths due to Posa [7].

Let $P = (v_1, v_2, \dots, v_m)$ be a path of maximal length in G . If v_m is adjacent to v_i (abbreviated $v_i \sim v_m$) for some i , then another path of maximal length is given by $P' = (v_1, \dots, v_i, v_m, v_{m-1}, \dots, v_{i+1})$.

We say that P' is a *rotation* of P with *fixed endpoint* v_1 , *pivot* v_i and *broken edge* $v_i \sim v_{i+1}$. We can then rotate P' in a similar fashion to get a new path P'' of the same length, and so on.

For $t \geq 0$, let $S_t = \{v \in V(P) \setminus \{v_1\} : v \text{ is the endpoint of a path obtainable from } P \text{ by at most } t \text{ rotations with fixed endpoint } v_1, \text{ and all broken edges in } P\}$

PROPOSITION 3.1 [6] For $t \geq 0$, $|S_{t+1}| \geq \frac{1}{2}|N(S_t)| - \frac{3}{2}|S_t|$.

Let

$$\lambda = \frac{(\log \log n)^2}{7500 \log n (\log \log \log n)};$$

$$t_0 = \left\lceil \frac{\log 4\lambda^2 n}{2(\log(1/(2\lambda)) - 4) - \log \sqrt{7}} \right\rceil + 2.$$

By Corollary 2.4(b), as long as $|S_t| < \lambda^2 n$, then $|N(S_t)| > \frac{(\frac{1}{2}-4\lambda)^2}{3\lambda^2}|S_t|$, and thus by Proposition 3.1, $|S_{t+1}| > \frac{(\frac{1}{2}-4\lambda)^2}{6\lambda^2}|S_t| - \frac{3}{2}|S_t|$, which implies

$$\frac{|S_{t+1}|}{|S_t|} > \frac{(\frac{1}{2} - 4\lambda)^2}{7\lambda^2}.$$

In particular, using $\Delta/\delta < c < \sqrt{2}$, after at most $t_0 - 2$ steps we have that $|S_t| > \frac{\lambda^2 n \Delta^4}{\delta^4}$.

By Corollary 2.4(c) and Proposition 3.1 when taking one more step we will have

$$|S_{t+1}| \geq \frac{1}{2}|N(S_t)| - \frac{3}{2}|S_t| \geq \frac{1}{2}\left(\frac{n}{2} - |S_t|\right) - \frac{3}{2}|S_t| \geq \frac{n}{4} - 2|S_{t+1}|,$$

which implies $|S_{t+1}| \geq \frac{n}{12}$.

Let $Y = V \setminus (S_{t+1} \cup N(S_{t+1}))$, then $e(S_{t+1}, Y) = 0$. By Corollary 2.4(d), we have

$$|Y| < \frac{\lambda^2 n^2 \Delta^4 / \delta^4}{|S_{t+1}|} \leq \frac{12\lambda^2 n \Delta^4}{\delta^4}.$$

So, $|N(S_{t+1})| = n - |S_{t+1}| - |Y| > n - |S_{t+1}| - \frac{12\lambda^2 n \Delta^4}{\delta^4}$.

Again using Proposition 3.1, we get

$$\begin{aligned} |S_{t+2}| &\geq \frac{1}{2}|N(S_{t+1})| - \frac{3}{2}|S_{t+1}| \\ &\geq \frac{1}{2}\left(n - |S_{t+1}| - \frac{12\lambda^2 n \Delta^4}{\delta^4}\right) - \frac{3}{2}|S_{t+1}| \\ &= \frac{1}{2}\left(n - \frac{12\lambda^2 n \Delta^4}{\delta^4}\right) - 2|S_{t+1}| \\ &> \frac{1}{2}n(1 - 48\lambda^2) - 2|S_{t+2}|. \end{aligned}$$

So

$$|S_{t+2}| > \frac{1}{6}n(1 - 48\lambda^2) > \frac{n}{7}, \text{ i.e. } |S_{t_0}| > \frac{n}{7}.$$

Let $B(v_1) = S_{t_0}$ and $A_0 = B(v_1) \cup \{v_1\}$. For each $v \in B(v_1)$ we can repeat the above argument to get $B(v)$, $|B(v)| > n/7$, of endpoints of maximum length paths with endpoint v . Note that each endpoint in $B(v)$ was obtained by at most $2t_0$ rotations of P . So, for each $a \in A_0$, $b \in B(a)$ there is a maximum length path $P(a, b)$ joining a and b which is obtainable from P by at most $\rho = 2t_0$ rotations.

We return to the initial path P and directed it. Since each endpoint in $B(v_1)$ is in P , we see $|P| \geq |B(v_1)| > n/7$. Then we can divide the path P into 2ρ disjoint segments $I_1, \dots, I_{2\rho}$ each of length at least $\lfloor n/14\rho \rfloor$. Since each path $P(a, b)$ is obtainable from P by at most ρ rotations there are at least ρ of the segments untouched (but possibly traversed in the opposite direction). we call each such segment *unbroken* in $P(a, b)$. These segments have an absolute orientation induced by P , and another, relative to this by $P(a, b)$ (where we direct that path from a to b).

Let

$$k = 2 \max\{1, \lceil 3000\rho\lambda \rceil\}.$$

We consider sequence $\sigma = I_{i_1}, \dots, I_{i_k}$ of k unbroken segments of P which occur in this order in $P(a, b)$, where σ specifies not only the order of segments in $P(a, b)$ but also their relative orientation. We say then that $P(a, b)$ contains σ . Note that as $P(a, b)$ has at least $\binom{\rho}{k}$ sequences σ .

For a given σ we denote by $L(\sigma)$ the set of all pairs $a \in A_0$, $b \in B(a)$, for which the path $P(a, b)$ contains σ . The total number of possible sequences σ is at most $(2\rho)_k 2^k$. Therefore by averaging we obtain that there exists a sequence σ_0 for which

$$L(\sigma_0) \geq \frac{n^2}{49} \frac{\binom{\rho}{k}}{(2\rho)_k 2^k} > \frac{n^2}{49} \left(\frac{\rho - k}{2\rho - k} \right)^k \frac{1}{k! 2^k}.$$

It is easy to check that $k \leq \rho/2$ when n sufficiently large. Then $(\rho - k)/(2\rho - k) \geq 1/3$, and it follows that there exists a sequence σ_0 for which $|L(\sigma_0)| \geq n^2/(49k!6^k)$. We fix such a sequence and denote

$$\alpha = \frac{1}{49k!6^k}.$$

Let $\hat{A} = \{a \in A_0 : L(\sigma_0) \text{ contains at least } \alpha n/2 \text{ pairs with } a \text{ as the first element}\}$. Then $|\hat{A}| \geq \alpha n/2$. For each $a \in \hat{A}$, let $\hat{B}(a) = \{b \in B(a) : (a, b) \in L(\sigma_0)\}$. The definition of \hat{A} guarantees that $|\hat{B}| \geq \alpha n/2$

Let C_1 be the union of the first $k/2$ segments of σ_0 , in the fixed order and with the fixed relative orientation in which they occur along any of the paths $P(a, b)$, $(a, b) \in L(\sigma_0)$. Let C_2 be the union of the last $k/2$ segments of σ_0 . Note that for $i = 1, 2$,

$$|C_i| \geq \frac{k}{2} \left\lfloor \frac{n}{14\rho} \right\rfloor \geq 3000\rho\lambda \left\lfloor \frac{n}{14\rho} \right\rfloor > 200n\lambda. \quad (3.1)$$

Given a path P and a set $S \subset V(P)$, a vertex $v \in S$ is called an *interior point* of S with respect to P if both neighbors of v along P are in S . The set of all interior points of S will be denoted by $\text{int}(S)$.

PROPOSITION 3.2 *The set C_1 contains a subset C'_1 with $|\text{int}(C'_1)| \geq nk/(60\rho)$ so that every vertex $v \in C'_1$ has at least $48\lambda d$ neighbors in $\text{int}(C'_1)$. A similar statement holds for C_2 .*

Proof: We start with $C'_1 = C_1$ and as long as there exists a vertex $v_j \in C'_1$ for which has less than $48\lambda d$ neighbors in $\text{int}(C'_1)$, we delete v_j and repeat. If this procedure continued for $r = |C_1|/8$ steps then we get a subset $R = \{v_1, v_2, \dots, v_r\}$, so that

$$|\text{int}(C'_1)| \geq |\text{int}(C_1)| - 3r = (1 - o(1))|C_1| - 3r > |C_1|/2 \geq nk/(60\rho)$$

and

$$e(R, \text{int}(C'_1)) \leq 48\lambda dr = 6\lambda d|C_1|. \quad (3.2)$$

But according to Corollary 2.2 and (3.1),

$$\begin{aligned} e(R, \text{int}(C'_1)) &\geq \frac{\delta^2}{nd}|R||\text{int}(C'_1)| - \frac{\lambda\Delta^2}{nd}\sqrt{|R|(n-|R|)|\text{int}(C'_1)|(n-|\text{int}(C'_1)|)} \\ &\geq \frac{\delta^2}{nd}|R||\text{int}(C'_1)| - \frac{\lambda\Delta^2}{d}\sqrt{|R||\text{int}(C'_1)|} \\ &\geq \frac{\delta^2}{nd}\frac{|C_1|^2}{16} - \frac{\lambda\Delta^2}{d}\sqrt{\frac{|C_1|^2}{16}} > \frac{\delta^2}{nd}\frac{200n\lambda|C_1|}{16} - \frac{\lambda\Delta^2|C_1|}{4d} \\ &= \lambda d|C_1|\left(\frac{25\delta^2}{2d^2} - \frac{\Delta^2}{4d^2}\right) \geq \lambda d|C_1|\left(\frac{25\delta^2}{2\Delta^2} - \frac{1}{4}\right) \\ &> 6\lambda d|C_1|. \end{aligned}$$

using that $\delta^2/\Delta^2 > 1/2$ in going to the last line, which is contradiction to (3.2). So, the result follows. \blacksquare

We fix the obtained sets C'_1 and C'_2 .

PROPOSITION 3.3 *There is a vertex $\hat{a} \in \hat{A}$ connected by an edge to $\text{int}(C'_1)$. Similarly there is a vertex $\hat{b} \in \hat{B}(\hat{a})$ connected by an edge to $\text{int}(C'_2)$.*

Proof: Recall that $|\hat{A}| \geq \frac{\alpha n}{2}$, and $|\text{int}(C'_1)| \geq nk/(60\rho)$. Therefore, by Corollary 2.4(d), the claim will follow if we will show that $\frac{\alpha n}{2} \frac{nk}{60\rho} \gg \frac{\Delta^4 \lambda^2 n^2}{\delta^4}$, or (substituting the value of α) $\frac{\delta^4}{\Delta^4 \lambda^2 \rho} \gg 5880(k-1)!6^k$.

Consider first the case $3000\rho\lambda \geq 1$. In this case,

$$\begin{aligned}
k &= 2(1 + o(1))3000\lambda\rho = 6000(1 + o(1))\frac{\lambda \log \lambda^2 n}{\log(1/\lambda)} \\
&\leq 6000(1 + o(1)) \left(-2\lambda + \frac{\log n}{(1/\lambda) \log(1/\lambda)} \right) \\
&= 6000(1 + o(1)) \frac{\log n}{\frac{7500 \log n (\log \log \log n)}{(\log \log n)^2} \log \log n} \\
&= \frac{0.8(1 + o(1)) \log \log n}{\log \log \log n},
\end{aligned}$$

and thus $5880(k-1)!6^k < (\log n)^{0.9}$. On the other hand, as $\delta/\Delta > 1/\sqrt{2}$,

$$\begin{aligned}
\frac{\delta^4}{\Delta^4 \lambda^2 \rho} &\geq \frac{1}{4\lambda^2} \frac{\log(1/\lambda)}{(1 + o(1)) \log \lambda^2 n} \\
&\geq \frac{\log^2 n (\log \log \log n)^2}{(\log \log n)^4} \frac{\log \log n}{(1 + o(1)) \log n} \\
&> \frac{(1 + o(1)) \log n}{(\log \log n)^3} \\
&\gg (\log n)^{0.9},
\end{aligned}$$

as required.

In the second case, $3000\rho\lambda < 1$, we get $k = 2$, then the expression $(k-1)!6^k$ is an absolute constant, while $\frac{\delta^4}{\Delta^4 \lambda^2 \rho} \geq \frac{1}{4} \frac{1}{\rho\lambda} \frac{1}{\lambda} \geq \frac{750}{\lambda} \rightarrow \infty$. The Proposition follows. \blacksquare

Now, let x be a vertex separating C'_1 and C'_2 along $P(\hat{a}, \hat{b})$, we consider two half path P_1 and P_2 obtained by splitting $P(\hat{a}, \hat{b})$ at x . Consider P_1 firstly. Let $T_i = \{v \in C'_1 \setminus \{x\} : v \text{ is the endpoint of a path obtainable from } P_1 \text{ by } i \text{ rotations with fixed endpoint } x, \text{ all pivots in } \text{int}(C'_1) \text{ and all broken edges in } P_1\}$.

PROPOSITION 3.4 *There exists an i for which $|T_i| \geq \lambda n (\Delta/\delta)^2$.*

Proof: It is enough to prove that there exists a sequence of sets $U_i \subseteq T_i$ such that $U_1 = 1$ and $|U_{i+1}| = 2|U_i|$, as long as $|U_i| < \lambda n (\Delta/\delta)^2$. Note that according to Proposition 3.3 \hat{a} has a neighbor in $\text{int}(C'_1)$, and therefore $T_1 \neq \emptyset$. Note also that if we perform a rotation a vertex from $\text{int}(C'_1)$ and broken edge in P_1 , then the resulting endpoint is in C'_1 .

Suppose we have found sets U_1, \dots, U_i as state above, and still $|U_i| < \lambda n (\Delta/\delta)^2$. We first show that

$$|T_{i+1}| \geq \frac{1}{2} |N(U_i) \cap \text{int}(C'_1)| - \frac{3}{2} \sum_{j=1}^i |U_j|.$$

Let

$$T = \{k \geq 1 : v_k \in N(U_i) \cap \text{int}(C'_1), v_{k-1}, v_k, v_{k+1} \notin \cup_{j=1}^i U_j\}.$$

Consider a vertex v_k with $k \in T$. Then v_k has a neighbor $w \in U_i$ which is also a interior vertex of C'_1 . So there exists a path Q with w as an endpoint, obtained from P_1 by i rotations with fixed endpoint x . As $v_{k-1}, v_k, v_{k+1} \notin \cup_{j=1}^i U_j$, both edges (v_{k-1}, v_k) and (v_k, v_{k+1}) are still present in Q . Rotating Q with a pivot v_k and one of the edges (v_{k-1}, v_k) and (v_k, v_{k+1}) as a broken edge will put one of v_{k-1}, v_{k+1} , say v_{k-1} in T_{i+1} . The only other vertex that possible cause v_{k-1} to be put into T_{i+1} is v_{k-2} if $k-2 \in T$. Therefore,

$$|T_{i+1}| \geq \frac{1}{2}|T| \geq \frac{1}{2}(|N(U_i) \cap \text{int}(C'_1)| - 3|\sum_{j=1}^i U_j|) \geq \frac{1}{2}|N(U_i) \cap \text{int}(C'_1)| - \frac{3}{2}\sum_{j=1}^i |U_j|.$$

As $\sum_{j=1}^i |U_j| < 2|U_i|$, the claim will follow if we prove that $|N(U_i) \cap \text{int}(C'_1)| \geq 10|U_i|$. Since $U_i \subset C'_1$, every vertex $u \in U_i$ has at least $48\lambda d$ neighbors in $\text{int}(C'_1)$. Therefore $e(U_i, \text{int}(C'_1)) \geq 48\lambda d|U_i|$. Let $W_i = N(U_i) \cap \text{int}(C'_1)$. If $|W_i| < 10|U_i|$, then by Corollary 2.2 we have

$$\begin{aligned} e(U_i, W_i) &\leq \frac{\Delta^2}{nd} \left(|U_i||W_i| + \lambda\sqrt{|U_i|(n-|U_i|)|W_i|(n-|W_i|)} \right) \\ &\leq \frac{\Delta^2}{nd}|U_i||W_i| + \frac{\lambda\Delta^2}{d}\sqrt{|U_i||W_i|} < \frac{10\Delta^2|U_i|^2}{nd} + \frac{\sqrt{10}\lambda\Delta^2|U_i|}{d} \\ &= \frac{\Delta^2}{d^2}\lambda d|U_i| \left(\frac{10|U_i|}{\lambda n} + \sqrt{10} \right) < \frac{\Delta^2}{\delta^2}\lambda d|U_i| \left(\frac{10\lambda n(\Delta/\delta)^2}{\lambda n} + \sqrt{10} \right) \\ &< 2\lambda d|U_i|(20 + \sqrt{10}) < 48\lambda d|U_i|, \end{aligned}$$

a contradiction. Therefore $|W_i| \geq 10|U_i|$, as desired. ■

Hence, the set V_1 of endpoints of all rotations of P_1 has cardinality $|V_1| \geq \lambda n(\Delta/\delta)^2$. Similarly the set V_2 of endpoints of all rotations of P_2 also has cardinality $|V_2| \geq \lambda n(\Delta/\delta)^2$. Then, $|V_1||V_2| \geq (\Delta/\delta)^4 \lambda^2 n^2$, by Corollary 2.4(d) there is an edge connecting V_1 and V_2 and thus closing the cycle. As G is connected by Corollary 2.4(e), this cycle is a Hamilton cycle. This completes the proof of Theorem 1.1. ■

References

- [1] S. Butler and F. Chung, Small spectral gap in the combinatorial Laplacian implies Hamiltonian, *Annals Comb.*, 13(2010), 403-412.
- [2] F. Chung, Discrete isoperimetric inequalities, *In: Surveys in Differential Geometry*, vol. IX, pp. 53-82, International Press, Somerville, 2004.
- [3] M. Fiedler, V. Nikiforov, Spectral radius and Hamiltonicity of graphs, *Linear Algebra Appl.*, 432(2010), 2170-2173.
- [4] R. M. Karp, Reducibility among combinatorial problems, In: R. E. Miller and J. W. Thatcher (eds.) *Comoplexity of Computer Computations*, pp. 85-103, Plenum, New York, 1972.

- [5] J. Komlós, E. Szemerédi, Hamilton cycles in random graphs, in: *Infinite and Finite sets*, vol. II, pp. 1003-1010, North-Holland, Amsterdam, 1975.
- [6] M. Krivelevich and B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, *J. Graph Theory*, 42(2003), 17-33.
- [7] L. Posa, Hamiltonian circuits in random graphs, *Discrete Math.*, 14(1976), 359-364.
- [8] M. Radcliffe, Sufficient Spectral Conditions for Hamiltonicity, <http://www.math.ucsd.edu>, 2011.
- [9] I. Tomescu, On Hamilton-connected regular Graphs, *J. Graph Theory*, 7(1983), 429-436.
- [10] B. Zhou, Signless Laplacian spectral radius and Hamiltonicity, *Linear Algebra Appl.*, 432(2010), 566-570.
- [11] G. D. Yu, Y. Z. Fan, Spectral conditions for a graph to be Hamilton-connected, arXiv:1207.6447v1.